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2D Surface Quasi-Geostrophic (SQG) Equations and its Regularity

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Abstract:
A detailed survey indicates that a gap exists in the numerical as well as the analytical study of 2D surface quasi geostrophic equations. Recently there is an increasing interest in its study as it is important in the field of fluid dynamics and similar to 3D Euler equations and hence with Navier Stokes equations, a millennium problem. In this paper, a review of the selected findings in the chronological order is presented. We present both the analytic and numerical results for the dissipative and inviscid cases.

Keywords: Surface quasi geostrophic equations, Inviscid, Dissipative, Global regularity, Finite time singularity

1. Introduction

The 2D Surface Quasi Geostrophic (SQG) equation is

$$\begin{aligned} \partial_t \theta + u.\nabla\theta + k(-\Delta)^{\alpha}\theta &= 0 \\ \nabla.u &= 0 \\ \theta(x,0) &= \theta_0(x) \end{aligned} \tag{1}$$

where $k \geq 0$ and $\alpha > 0$ are parameters, $\theta = \theta(x_1, x_2, t)$ is a scalar representing the potential temperature and $u = (u_1, u_2)$ is the velocity field determined from θ by the stream function ψ with the auxiliary relations

$$(u_1, u_2) = (-\partial_{x_2}\psi, \partial_{x_1}\psi),$$
$$(-\Delta)^{1/2}\psi = \theta.$$

Assuming $\Lambda = (-\Delta)^{1/2}$ and $\nabla^{\perp} = (-\partial_{x_2}, \partial_{x_1})$, the above relation can be written as

$$u = \nabla^{\perp}\Lambda^{-1}\theta = (-R_2\theta, R_1\theta)$$

where R_1 and R_2 are the usual Riesz transforms. The spatial domains concerned here is the periodic box T^2 or R^2. Depending upon k and α, the equation can be divided into the following categories:

1. When $\kappa = 0$, the equation (1.1) is called the inviscid SQG equation.
2. When $\kappa > 0$, the equation (1.1) is called the dissipative SQG equation.
   (a) When $\alpha > \frac{1}{2}$ the equation (1.1) is called the subcritical SQG equation.
   (b) When $\alpha = \frac{1}{2}$ the equation (1.1) is called the critical SQG equation.
   (c) When $\alpha < \frac{1}{2}$ the equation (1.1) is called the supercritical SQG equation.

The 3D Euler equation is given by

$$\partial_t u + u.\nabla u = -\frac{\nabla P}{\rho} + g, \ \ \nabla.u = 0 \tag{2}$$

where u, the velocity field; P, the fluid pressure and ρ, fluid density.
Similarly, the 3D Navier Stokes equation is given by

$$\partial_t u + u.\nabla u = -\frac{\nabla P}{\rho} + \nu\nabla^2 u, \ \nabla.u = 0 \tag{3}$$

The 2D inviscid SQG and 3D Euler equations share several common features which will be shown later. Similarly, there are some similarities between the 3D Euler Equations and 3D Navier Stokes Equations. So we pursue this review to reveal the current status of the SQG equation so that we may know about the possibility of further extension of the existing results to the 3D Euler and hence the Navier Stokes equations.

The purpose of this expository paper is to review the current status of the 2D Surface Quasi Geostrophic equations and explore it with researchers in the field of fluid dynamics. The review topics are limited to the global uniqueness and existence of the solutions of the equations for $0 \leq \alpha, \kappa$ in time and space, both analytically and numerically. Since the problem is similar to the millennium problem and regularity criteria for this can be extended further to complex situations, this review will be helpful to explore the situations and further extensions.

In the first part of this review, we present the selected analytical results, and in the second part, we present some numerical results.

2. Previous Works

The general 3D quasi geostrophic equations, first derived by J.G. Charney in 1940's. These equations have been very successful in describing the major features of large-scale motions in the atmosphere and the oceans in the midlatitudes [1]. The inviscid SQG equation is an important example of an active scalar. Also, this is an important testbed for turbulence because of its distinctive feature [2, 3].

The authors in [4] studied the formation of strong and potentially singular fronts in a two dimensional quasi geostrophic active scalar through the symbiotic interaction of mathematical theory and numerical experiments. They revealed the formation of geophysical flows in the atmosphere and issues of frontogenesis, formation of strong fronts between the masses of hot and cold air within quasi geostrophic approximations without explicitly incorporating ageostrophic effects. They developed the physical and mathematical analogies between the 2D inviscid SQG and the 3D Euler equations for the incompressible flow.

The quantities $\nabla^{\perp}\theta$ and $\omega = \nabla \times v$ play the similar role in two different situation, one in SQG and the second one in the 3D Euler equations respectively. Also, there is geometric analogy between the level sets of θ for 2D active scalar and the vortex lines of 3D Euler equations. The 3D incompressible Euler equations in vorticity-stream function is given by

$$\frac{D\omega}{Dt} = (\nabla v)\omega \tag{4}$$

where $\frac{D}{Dt} = \frac{\partial}{\partial t}, v = (v_1, v_2, v_3)$ is the three dimensional velocity field with div $v = 0$ and $\omega = \nabla \times v$ is the vorticity vector.

The 2D QG active scalar is given by the equation

$$\frac{D\theta}{Dt} = \frac{\partial \theta}{\partial t} + v.\nabla\theta = 0 \tag{5}$$

with the two dimensional velocity $v = (v_1, v_2)$ is determined from θ by a stream function ψ is given by $(v_1, v_2) = (-\psi_{x_2}, \psi_{x_1})$. Differentiating equation (2.2), we have

$$\frac{D\nabla^{\perp}\theta}{Dt} = (\nabla v)\nabla^{\perp}\theta. \tag{6}$$

Here the velocity v in (2.1) is determined from the vorticity ω by the Biot-Savart law and the strain matrix S, the symmetric part of the velocity gradient. The right hand side of (2.1) can be written in terms of S and ω.

For the 2D SQG active scalar, the velocity field is given by

$$v = \nabla^{\perp}\psi = -\int_{R^2} \frac{1}{|y|}\nabla^{\perp}\theta(x+y)dy. \tag{7}$$

The velocity in terms of ω is given by

$$v = \int_{R^d} K_d(y)\omega(x+y)dy \tag{8}$$

where $K_d(y)$ is homogeneous of degree $1-d$ in R^d for $d = 2, 3$.

Thus with these $\nabla^{\perp}\theta$ and vorticity, the evolution equation for $\nabla^{\perp}\theta$ from (2.3) has completely parallel analytic structure in 2D as the equation of evolution of vorticity, ω, in (2.1) for the 3D incompressible flow. The infinitesimal length for vortex line, whose magnitude is ω, is given by

$$\frac{D|\omega|}{Dt} = \alpha|\omega| \tag{9}$$

with $\alpha(x,t) = S(x,t)\xi.\xi$, where S is the symmetric matrix. Again the infinitesimal length of level set for θ is given by $|\nabla^{\perp}\theta|$ and evolution equation for the infinitesimal arc length is given by

$$\frac{D|\nabla^{\perp}\theta|}{Dt} = \alpha|\nabla^{\perp}\theta| \tag{10}$$

with $\alpha(x,t) = S(x,t)\xi.\xi$ and $\xi = \frac{\nabla^{\perp}\theta}{|\nabla^{\perp}\theta|}$. Since the two level sets equations for the two situations in equations (2.6) and (2.7) have the similar structure, the level sets of the solutions of the quasi geostrophic active scalar seem to correspond to vortex lines in the 3D Euler equations. The authors in [4] developed the mathematical criterion to characterize how the smooth solution of the equation (2.2) can be singular. This is the simplest type which is analog to characterize the singular solution of 3D Euler equation in [5] which is stated as:

"*The time interval* $[0,T*]$ *with* $T* < \infty$ *is a maximal interval of a smooth solution for the* 2*D quasi geostrophic active scalar if and only if* $\int_0^T |\nabla\theta|_{L^\infty}(s)ds \rightarrow \infty$ *as* $T \rightarrow T*$ *with norm* $|f|_{L^\infty} = max_{x\in R^2}|f(x)|$" [4].

Majda and Tabak, in 1996, found that the 2D Euler equation and SQG model have fronts with different behavior: the first one end up growing linearly due to the velocity field created nonlocally whereas the second gives a sustained nonlinear steepening of fronts. Also, their numerical studies showed that 2D Euler gives rise to fast-growing fronts switching nearly a linear regime whereas SQG fronts exhibit at a slower rate initially and then sustain a long nonlinear self-stretching process ending up with finite time collapse [6].

In 1997, Wu considered inviscid limits for both the smooth and weak solutions for the 2D dissipative QGS equations and established that the classical solutions of dissipative equations with smooth initial data tend to the solutions of the corresponding non dissipative equations when the dissipative coefficient tends to zero. The convergence is in strong L^2 sense. The methods used by Foias-Temam [7] and Doering-Titi [8] for the NS equations were used to establish exponential decay of the spatial Fourier spectrum for the solutions of the dissipative quasi geostrophic equation with the consideration of the general norm and the different methods of treating nonlinear term [9].

In 2001, Wu established global regularity results for the regularized models with critical or subcritical indices. Also, the proof of Onsager's conjecture [10] concerning weak solutions of 3D Euler equations and the notion of dissipative solution of Duchon and Robert [11] were extended to the weak solution of the quasi geostrophic equations [12].

Constantin, Cordoba and Wu, in 2001, proved the existence and uniqueness of global classical solutions of the critical dissipative quasi geostrophic equation for the initial data that have small L^∞- norm. Here the importance of an L^∞ smallness condition is due to fact that L^∞ is a conserved norm for the non dissipative quasi geostrophic equation [13].

Cordoba and Feffermen, in 2002, established that the distance between the two level curves cannot decrease faster than a double exponential time. This collapse assumption weakens the assumptions made in [4] for the classical frontogenesis and the simple hyperbolic saddle in [14]. They discussed two equations, Quasi Geostrophic equations and Two Dimensional Euler Equations, having common property that a scalar function is convected by flow. This implies that the level curves are transported by the flow [15].

A. Cordoba and D. Cordoba, in 2004, studied the initial value problem for dissipative 2D Quasi-geostrophic equations. They proved the local existence and global results for small initial data in the super-critical case. Also, they studied decay of L_p-norms and asymptotic behavior of viscosity solution in the critical case. Their studies were based on the maximum principle [16].

Wu, in 2005, established existence and uniqueness results for the 2D dissipative quasi geostrophic(QG) equations with the initial data in the Besov space or the space created by him which is the generalization of the Besov space, and focused on the critical or super critical fractional power of the Laplacian for which the dissipation is insufficient to balance the nonlinearity [17].

Kiselev, Nazarov and Volberg, in 2007, constructed a special family of moduli of continuity that are preserved by the dissipative evolution and obtained the estimate for $||\kappa\nabla\theta||_\infty$ and proved the following result:

Theorem 1 *The quasi geostrophic equation with periodic smooth initial data* $\theta_0(x)$ *has unique solution and the following estimate holds:*

$$||\nabla\theta||_\infty \leq C||\nabla\theta_0||_\infty \exp\exp\{C||\theta_0||_\infty\}.$$

The proof of theorem is for the global wellposedness for 2D SQG and they revealed that the result cannot be extended to Navier Stokes equations due to the structural difference in between them and this gave the global regularity for the general data for the periodic cases only [18].

Dong and Du, in 2008, studied the critical dissipative quasi-geostrophic equations in R^2 with arbitrary H^1 initial data and proved the global wellposedness result by adapting the method in [18] with a suitable modification and certain decay estimates. They also discussed the decay in time estimate for higher order homogeneous Sobolev norms of the solutions [19].

In 2008, Constantin and Wu established the following result:

Theorem 2 *If a Leray-Hopf weak solution for the* $2D$ *quasi geostrophic equation with* $\alpha < \frac{1}{2}$ *is Holder continuous* $\theta \in C^{\delta}(R^2)$ *with* $\delta > 1-2\alpha$ *on the interval* $[t_0,t]$ *then it is actually a classical solution on* $(t_0,t]$.

With the use of Little Paley Decomposition and Besov space techniques, the functions were represented in the Holder space. They showed that if $\theta \in C^{\delta}$ then it also belongs to the Besov space $B_{p,\infty}^{\delta(1-\frac{2}{p})}$ for $p \geq 2$ where Besov space $B^{s,p,\theta}(R^N)$ is the set of all functions $u \in L^1_{\text{loc}}(R^N)$ with $||u||_{B^{s,p,\theta}(R^N)} = ||u||_{L^p(R^N)} + |u|_{B^{s,p,\theta}(R^N)} < \infty$ where $1 \leq p,\theta \leq \infty$ and $0 < s < 1$. For the sufficiently large value of p, they showed that the same solution belongs to the space $C^{\delta_1} \cap B^{\delta_1}_{p,\delta}$ for $\delta_1 > 1-2\alpha$ and extended the solution to the space $C^{\delta_2} \cap B^{\delta_2}_{p,\delta}$ with $\delta_2 > \delta_1$. Using iteration, they showed that the solution belongs to C^{γ} with $\gamma > 1$ and confirmed that the solution is a classical one [20].

Dong and Pavlovic, in 2009, established a regularity criterion for weak solutions of the dissipative quasi geostrophic equation (with dissipation $(-\Delta)^{\frac{\gamma}{2}}, 0 < \gamma \leq 1$). More precisely they proved the following result:

Theorem 3 *If* $\theta \in L_t^{r_o}((0,T); B^{\alpha}_{p,\infty}(R^2))$ *with* $\alpha = \frac{2}{p} + 1 - \gamma + \frac{\gamma}{r_o}$ *is a weak solution of the 2D quasi geostrophic equation then* θ *is a classical solution in* $(0,T] \times R^2$.

They extended the regularity result of [20] to scaling invariant spaces [21]. Caffarelli and Vasseur, in 2010, showed that solution of quasi geostrophic equations with L^2 initial data and critical diffusion $(-\Delta)^2$ are locally smooth for any space dimension and they proved the global regularity for the general data for whole space [22].

3. Numerical Results

Constantin, Majda and Tabak, in 1994, performed the numerical experiments on a 2π-periodic box and predicted strong front formation and potential singular behavior of the solutions. They used spectral collocation method with an exponential filter which was basically the method developed by E and Shu [23, 24] for the incompressible flow with minor modifications. Their numerical method monitored two physical quantities, kinetic energy and the pseudo energy. They calculated $u(\theta)$ in the Fourier space and $u.\nabla\theta$ in the physical space and the time stepping through fourth order RK method. They used finer partitions ranging from 256^2 to 512^2 to 1024^2. The following three types of initial data were considered.

1. $\theta(x,0) = \sin x_1 \sin x_2 + \cos x_2$
2. $\theta(x,0) = -(\cos 2x_1 \cos x_2 + \sin x_1 \sin x_2)$
3. $\theta(x,0) = \cos 2x_1 \cos x_2 + \sin x_1 \sin x_2 + \cos 2x_1 \sin 3x_2$

The first initial condition was considered as the simplest type of smooth initial data with nonlinear behavior and also the combination of two lowest eigenmodes.

The first data set involves a hyperbolic saddle in the initial level sets of temperature in the regime of strong nonlinear behavior. The numerical solutions indicate strong nonlinear front formation and potentially singular behavior. The second data set involves the elliptic level sets in θ and the numerical solutions asserts that the solution behaves nonlinearly as in the first set initially but self consistently saturates to exponential growths of gradients without singular behavior. In the third set the more general initial condition is considered which also indicates the robust feature of strong front formation. They concluded that
"if the geometry of level sets of the active scalar is simple and does not contain a hyperbolic saddle in the region of strongly nonlinear behavior, then no singular behavior is possible"[4]. Figure 1 shows the evolution of level sets and 3D surface plot of the type (3) data.

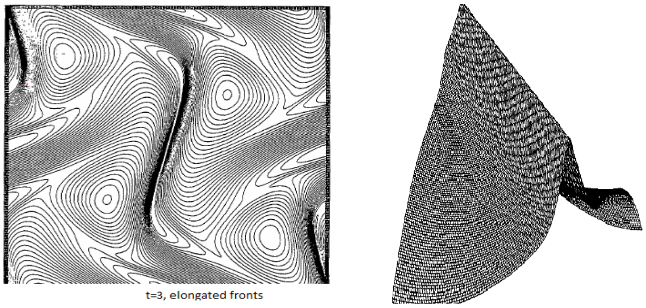

Figure 1: Contour plot and Nature of front

Ohkitani and Yamada, in 1997, based on the simple initial condition used in [4], proposed that the temperature gradient can be fitted equally well by a double-exponential function of time rather than an algebraic blow up. Also for the viscous case, a comparison was made between a series of computations with different Reynolds number. Their findings revealed that the time at which the temperature gradient reaches the first local maximum is dependent on the Reynolds number in a double logarithmic manner. This implies that the inviscid flow is globally regular [25].

In 1998, D. Cordoba confirmed of singular behavior of the solution of inviscid SQG when the level sets contain hyperbolic saddle. He showed that simple hyperbolic saddle breakdown cannot occur in finite time. In addition, the author demonstrated that the angle of the saddle cannot approach closure in a finite amount of time and cannot increase faster than a double exponential function of time. This implied the equivalent outcomes hold valid for the incompressible 2D and 3D Euler equations [14].

Ohkitani and Sakajo, in 2012, studied numerically the long-time evolution of the surface quasi-geostrophic equation with generalized viscosity of the form $(-\Delta)^{\alpha}$, where global regularity has been proved mathematically for the subcritical parameter range $\alpha \geq \frac{1}{2}$. In the supercritical range, their numerical findings indicated that smooth evolution continues, though with a gradual damping over time. They also discovered that the index $\alpha = \frac{1}{2}$ is not the critical case in their numerical study [26].

Constantin, Sharma, Wu et all, in 2012, used pseudospectral method with an improved exponential filter. They extended the work of [4] and revealed the nature of solution for the longer time interval. The Fourier space was used for computing the derivatives while the physical space was used for computing the products. To perform time integration, the 4th order Runge Kutta method was utilized. For parallel computation, the pseudospectral algorithm was parallelized using slab decomposition. To reduce the aliasing error, they used the exponential filter $f(x) = \exp(-\alpha x^m)$ where $\alpha = 36, m = 19$ where $\alpha = -\log \varepsilon, \varepsilon$ being the order of machine precision. This technique aids in suppressing the higher frequency modes of $1/3$, while leaving the modes of $2/3$ unchanged. The exponential filter they used is shown in Figure 2.

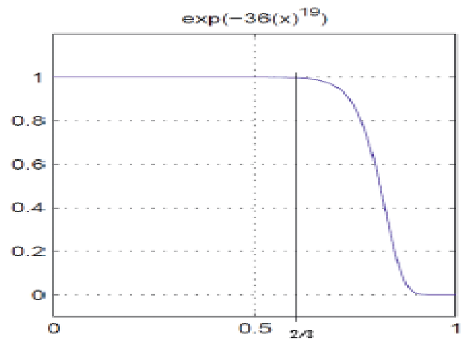

Figure 2: Exponential filter

The researchers made a prediction about the emergence of a robust hyperbolic saddle front around $t = 7.5$, which was previously observed in [4]. This was followed by a sharp antiparallel double front while the maximum gradient continued to increase until approximately $t = 13.5$. Afterward, the gradient decayed without any regeneration of strong fronts.One of the graphs for the gradient growth is presented in Figure 3.

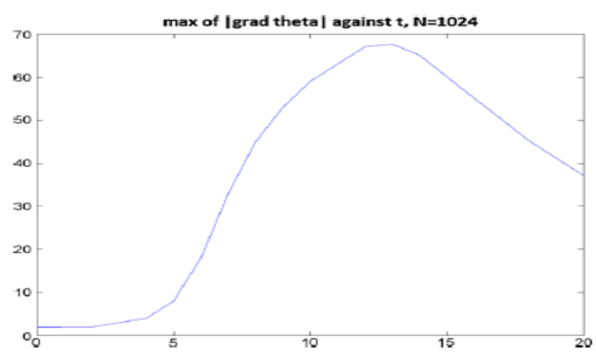

Figure 3: Gradient growth

They revealed that numerically there was no evidence of critical behaviour at $\alpha = \frac{1}{2}$ which agrees with the study of [26]. Their numerical computation monitored on the growth of L^2- norm and helicity. They used larger value of N for more finer partition [27]. The evolution of level curves of some of their numerical simulations are presented in Figure 4 and Figure 5.

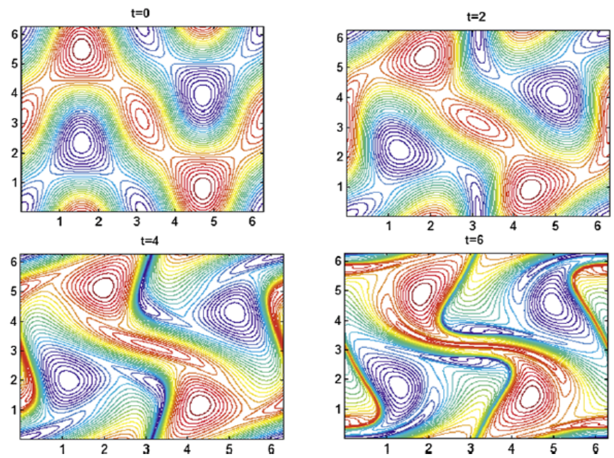

Figure 4: Contour plots of θ

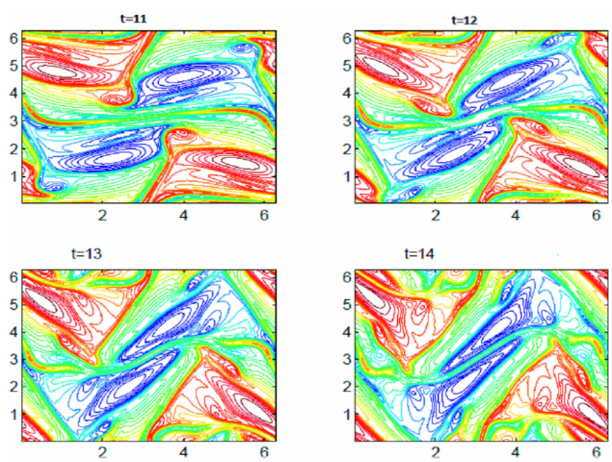

Figure 5: Contour plots of θ

4. Conclusions and Recommendations

4.1 Conclusions

We have reviewed the current state of the SQG equations in order to determine the feasibility of extending the current findings to the 3D Euler equations, and consequently, the Navier-Stokes equations. We have examined both analytical and numerical outcomes for both the viscous and inviscid cases of the SQG equations. Furthermore, we have directed our attention towards the regularity of the supercritical cases, which remains an open problem in the field of fluid dynamics.

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